Model Answer of Final Exam

Answer of Question (1)

[a]

Let A_1 be the event "4, 5 or 6 on first toss," and A_2 be the event "1, 2, 3 or 4 on second toss." Then we are looking for $P(A_1 \cap A_2)$.

Method 1.

$$P(A_1 \cap A_2) = P(A_1)P(A_2 \mid A_1) = P(A_1)P(A_2) = \left(\frac{3}{6}\right)\left(\frac{4}{6}\right) = \frac{1}{3}$$

We have used here the fact that the result of the second toss is *independent* of the first so that $P(A_2 \mid A_1) = P(A_2)$. Also we have used $P(A_1) = 3/6$ (since 4, 5 or 6 are 3 out of 6 equally likely possibilities) and $P(A_2) = 4/6$ (since 1, 2, 3 or 4 are 4 out of 6 equally likely possibilities).

Method 2.

Each of the 6 ways in which a die can fall on the first toss can be associated with each of the 6 ways in which it can fall on the second toss, a total of $6 \cdot 6 = 36$ ways, all equally likely.

Each of the 3 ways in which A_1 can occur can be associated with each of the 4 ways in which A_2 can occur to give $3 \cdot 4 = 12$ ways in which both A_1 and A_2 can occur. Then

$$P(A_1 \cap A_2) = \frac{12}{36} = \frac{1}{3}$$

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[b]

(*a*)

$$\begin{aligned} P(1) &= P(1 \cap H \text{ or } 1 \cap S \text{ or } 1 \cap D \text{ or } 1 \cap C) \\ &= P(1 \cap H) + P(1 \cap S) + P(1 \cap D) + P(1 \cap C) \\ &= \frac{1}{52} + \frac{1}{52} + \frac{1}{52} + \frac{1}{52} = \frac{1}{13} \end{aligned}$$

This could also have been achieved from the sample space of Problem 1.10(*a*) where each sample point, in particular "ace," has probability 1/13. It could also have been arrived at by simply reasoning that there are 13 numbers and so each has probability 1/13 of being drawn.

(*b*) $P(11 \cap H) = \dfrac{1}{52}$

(*c*) $P(3 \cap C \text{ or } 6 \cap D) = P(3 \cap C) + P(6 \cap D) = \dfrac{1}{52} + \dfrac{1}{52} = \dfrac{1}{26}$

(*d*) $P(H) = P(1 \cap H \text{ or } 2 \cap H \text{ or } \ldots 13 \cap H) = \dfrac{1}{52} + \dfrac{1}{52} + \cdots + \dfrac{1}{52} = \dfrac{13}{52} = \dfrac{1}{4}$

This could also have been arrived at by noting that there are four suits and each has equal probability $\frac{1}{4}$ of being drawn.

(*e*) $P(H') = 1 - P(H) = 1 - \dfrac{1}{4} = \dfrac{3}{4}$ using part (*d*) and Theorem 1-17, page 6.

(*f*) Since 10 and S are not mutually exclusive we have from Theorem 1-19

$$P(10 \cup S) = P(10) + P(S) - P(10 \cap S) = \frac{1}{13} + \frac{1}{4} - \frac{1}{52} = \frac{4}{13}$$

(*g*) The probability of neither four nor club can be denoted by $P(4' \cap C')$. But by Theorem 1-12(*a*), page 3, $4' \cap C' = (4 \cup C)'$. Thus

$$\begin{aligned} P(4' \cap C') &= P[(4 \cup C)'] = 1 - P(4 \cup C) \\ &= 1 - [P(4) + P(C) - P(4 \cap C)] \\ &= 1 - \left[\frac{1}{13} + \frac{1}{4} - \frac{1}{52}\right] = \frac{9}{13} \end{aligned}$$

Answer of Question (2)

[a]

(*a*) We must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$, i.e.

$$\int_{-\infty}^{\infty} \frac{c\,dx}{x^2+1} = c\tan^{-1} x \Big|_{-\infty}^{\infty} = c\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1$$

so that $c = 1/\pi$.

(*b*) If $\frac{1}{3} \leqq X^2 \leqq 1$, then either $\frac{\sqrt{3}}{3} \leqq X \leqq 1$ or $-1 \leqq X \leqq -\frac{\sqrt{3}}{3}$. Thus the required probability is

$$\frac{1}{\pi}\int_{-1}^{-\sqrt{3}/3} \frac{dx}{x^2+1} + \frac{1}{\pi}\int_{\sqrt{3}/3}^{1} \frac{dx}{x^2+1} = \frac{2}{\pi}\int_{\sqrt{3}/3}^{1} \frac{dx}{x^2+1}$$

$$= \frac{2}{\pi}\left[\tan^{-1}(1) - \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)\right]$$

$$= \frac{2}{\pi}\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{1}{6}$$

[b]

(*a*) The sample points (x, y) for which probabilities are different from zero are indicated in Fig. 2-14. The probabilities associated with these points, given by $c(2x+y)$, are shown in Table 2-6. Since the grand total, $42c$, must equal 1, we have $c = 1/42$.

Table 2-6

X \ Y	0	1	2	3	Totals ↓
0	0	c	$2c$	$3c$	$6c$
1	$2c$	$3c$	$4c$	$5c$	$14c$
2	$4c$	$5c$	$6c$	$7c$	$22c$
Totals →	$6c$	$9c$	$12c$	$15c$	$42c$

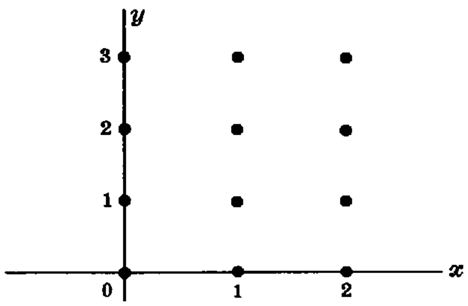

Fig. 2-14

(*b*) From Table 2-6 we see that

$$P(X=2, Y=1) = 5c = \frac{5}{42}$$

(*c*) From Table 2-6 we see that

$$P(X \geqq 1, Y \leqq 2) = \sum_{x \geqq 1}\sum_{y \leqq 2} f(x,y)$$

$$= (2c + 3c + 4c) + (4c + 5c + 6c)$$

$$= 24c = \frac{24}{42} = \frac{4}{7}$$

as indicated by the entries shown shaded in the table.

Answer of Question (3)

[a]

(*a*) $$E(X^*) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma}[E(X-\mu)] = \frac{1}{\sigma}[E(X)-\mu] = 0$$

since $E(X) = \mu$.

(*b*) $$\text{Var}(X^*) = \text{Var}\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2}E[(X-\mu)^2] = 1$$

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[b]

(*a*) $$E(X) = \sum_x\sum_y x f(x,y) = \sum_x x\left[\sum_y f(x,y)\right]$$

$$= (0)(6c) + (1)(14c) + (2)(22c) = 58c = \frac{58}{42} = \frac{29}{21}$$

(*b*) $$E(Y) = \sum_x\sum_y y f(x,y) = \sum_y y\left[\sum_x f(x,y)\right]$$

$$= (0)(6c) + (1)(9c) + (2)(12c) + (3)(15c) = 78c = \frac{78}{42} = \frac{13}{7}$$

(*c*) $$E(XY) = \sum_x\sum_y xy\, f(x,y)$$

$$= (0)(0)(0) + (0)(1)(c) + (0)(2)(2c) + (0)(3)(3c)$$
$$+ (1)(0)(2c) + (1)(1)(3c) + (1)(2)(4c) + (1)(3)(5c)$$
$$+ (2)(0)(4c) + (2)(1)(5c) + (2)(2)(6c) + (2)(3)(7c)$$

$$= 102c = \frac{102}{42} = \frac{17}{7}$$

(*d*) $$E(X^2) = \sum_x\sum_y x^2 f(x,y) = \sum_x x^2\left[\sum_y f(x,y)\right]$$

$$= (0)^2(6c) + (1)^2(14c) + (2)^2(22c) = 102c = \frac{102}{42} = \frac{17}{7}$$

(*e*) $$E(Y^2) = \sum_x\sum_y y^2 f(x,y) = \sum_y y^2\left[\sum_x f(x,y)\right]$$

$$= (0)^2(6c) + (1)^2(9c) + (2)^2(12c) + (3)^2(15c) = 192c = \frac{192}{42} = \frac{32}{7}$$

(*f*) $$\sigma_X^2 = \text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{17}{7} - \left(\frac{29}{21}\right)^2 = \frac{230}{441}$$

(*g*) $$\sigma_Y^2 = \text{Var}(Y) = E(Y^2) - [E(Y)]^2 = \frac{32}{7} - \left(\frac{13}{7}\right)^2 = \frac{55}{49}$$

Answer of Question (4)

[a]

(*a*) Weights recorded as being between 120 and 155 lb can actually have any value from 119.5 to 155.5 lb, assuming they are recorded to the nearest pound.

$$119.5 \text{ lb in standard units} = (119.5 - 151)/15 = -2.10$$

$$155.5 \text{ lb in standard units} = (155.5 - 151)/15 = 0.30$$

$$\begin{aligned}\text{Required proportion of students} &= (\text{area between } z = -2.10 \text{ and } z = 0.30) \\ &= (\text{area between } z = -2.10 \text{ and } z = 0) \\ &\quad + (\text{area between } z = 0 \text{ and } z = 0.30) \\ &= 0.4821 + 0.1179 = 0.6000\end{aligned}$$

Then the number of students weighing between 120 and 155 lb is $500(0.6000) = 300$.

(*b*) Students weighing more than 185 lb must weigh at least 185.5 lb.

$$185.5 \text{ lb in standard units} = (185.5 - 151)/15 = 2.30$$

Required proportion of students

$$\begin{aligned}&= (\text{area to right of } z = 2.30) \\ &= (\text{area to right of } z = 0) \\ &\quad - (\text{area between } z = 0 \text{ and } z = 2.30) \\ &= 0.5 - 0.4893 = 0.0107\end{aligned}$$

Then the number of students weighing more than 185 lb is $500(0.0107) = 5$.

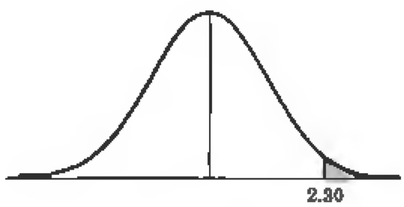

Fig. 4-13

If W denotes the weight of a student chosen at random, we can summarize the above results in terms of probability by writing

$$P(119.5 \leqq W \leqq 155.5) = 0.6000 \qquad P(W \geqq 185.5) = 0.0107$$

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[b]

(*a*) Let X be the random variable giving the number of heads in 10 tosses. Then

$$P(X=3) = \binom{10}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^7 = \frac{15}{128} \qquad P(X=4) = \binom{10}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^6 = \frac{105}{512}$$

$$P(X=5) = \binom{10}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^5 = \frac{63}{256} \qquad P(X=6) = \binom{10}{6}\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^4 = \frac{105}{512}$$

Then the required probability is

$$P(3 \leqq X \leqq 6) = \frac{15}{128} + \frac{105}{512} + \frac{63}{256} + \frac{105}{512} = \frac{99}{128} = 0.7734$$

(b) The probability distribution for the number of heads in 10 tosses of the coin is shown graphically in Figures 4-15 and 4-16, where Fig. 4-16 treats the data as if they were continuous. The required probability is the sum of the areas of the shaded rectangles in Fig. 4-16 and can be approximated by the area under the corresponding normal curve, shown dashed. Treating the data as continuous, it follows that 3 to 6 heads can be considered as 2.5 to 6.5 heads. Also, the mean and variance for the binomial distribution are given by $\mu = np = 10(\frac{1}{2}) = 5$ and $\sigma = \sqrt{npq} = \sqrt{(10)(\frac{1}{2})(\frac{1}{2})} = 1.58$. Now

$$2.5 \text{ in standard units} = (2.5 - 5)/1.58 = -1.58$$

$$6.5 \text{ in standard units} = (6.5 - 5)/1.58 = 0.95$$

Required probability

$= $ (area between $z = -1.58$ and $z = 0.95$)

$= $ (area between $z = -1.58$ and $z = 0$)
$+$ (area between $z = 0$ and $z = 0.95$)

$= 0.4429 + 0.3289 = 0.7718$

which compares very well with the true value 0.7734 obtained in part (a). The accuracy is even better for larger values of n.

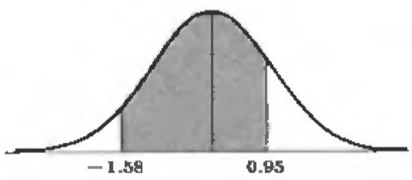

Fig. 4-17

Answer of Question (5)

The normal equations are

$$(1) \qquad \begin{aligned} \Sigma y &= an + b\Sigma x + c\Sigma x^2 \\ \Sigma xy &= a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \\ \Sigma x^2 y &= a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \end{aligned}$$

The work involved in computing the sums can be arranged as in Table 8-9.

Table 8-9

x	y	x^2	x^3	x^4	xy	x^2y
1.2	4.5	1.44	1.73	2.08	5.40	6.48
1.8	5.9	3.24	5.83	10.49	10.62	19.12
3.1	7.0	9.61	29.79	92.35	21.70	67.27
4.9	7.8	24.01	117.65	576.48	38.22	187.28
5.7	7.2	32.49	185.19	1055.58	41.04	233.93
7.1	6.8	50.41	357.91	2541.16	48.28	342.79
8.6	4.5	73.96	636.06	5470.12	38.70	332.82
9.8	2.7	96.04	941.19	9223.66	26.46	259.31
$\Sigma x =$ 42.2	$\Sigma y =$ 46.4	$\Sigma x^2 =$ 291.20	$\Sigma x^3 =$ 2275.35	$\Sigma x^4 =$ 18,971.92	$\Sigma xy =$ 230.42	$\Sigma x^2 y =$ 1449.00

Then the normal equations (1) become, since $n = 8$,

$$(2) \qquad \begin{aligned} 8a + 42.2\,b + 291.20\,c &= 46.4 \\ 42.2\,a + 291.20\,b + 2275.35\,c &= 230.42 \\ 291.20\,a + 2275.35\,b + 18971.92\,c &= 1449.00 \end{aligned}$$

Solving, $a = 2.588$, $b = 2.065$, $c = -0.2110$; hence the required least-squares parabola has the equation

$$y = 2.588 + 2.065\,x - 0.2110\,x^2$$